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APPROXIMATING FINANCIAL TIME SERIES WITH WAVELETS

Abstract

Financial time series show many characteristic properties including the phenomenon of clustering of variance, fat-tail distribution, and negative correlation between the rates of return and the volatility of their variance. These facts often render standard methods of parameter estimation and forecasting ineffective. An important feature of financial time series is that they can be characterized by long samples. This causes the models used for their estimation to potentially be more extensive.

The aim of the article is to use wavelets to approximate and predict a series. The article describes the author's model for financial time forecasting and provides basic information about wavelets necessary for proper understanding of the proposed wavelet algorithm. The algorithm uses a Daubechies wavelet.

Keywords: prediction, wavelets, wavelet transform, approximation.
JEL Classification: C10, C20, C40.

1. Wavelet Transform

The wavelet transform is the result of the transformation of the operand (the operator's argument, i.e. the function of a given space in itself) under the influence of the operator. Exemplary transforms include the Laplace transform, the Fourier transform, the wavelet transform, the Burrows-Wheeler transform and the Hilbert transform. Wavelet transform is used in this article. There are two ways to determine wavelet transform. Most transforms can only be done using a formula. For example, the Laplace transform of a time function $v(t)$ is calculated by the formula:

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$$V(s)=\int_{-\infty}^{+\infty} v(t)e^{-st}\,dt. \tag{1}$$

There is a similar formula for the wavelet transform and Fourier transform. The most important difference between Fourier transform and Laplace transform is described by Euler (1744): “While the Fourier transform of a function is a complex function of a real variable (frequency), the Laplace transform of a function is a complex function of a complex variable. Laplace transforms are usually restricted to functions of t with $t > 0$. A consequence of this restriction is that the Laplace transform of a function is a holomorphic function of the variable s. Unlike the Fourier transform, the Laplace transform of a distribution is generally a well-behaved function. Also techniques of complex variables can be used directly to study Laplace transforms. As a holomorphic function, the Laplace transform has a power series representation. This power series expresses a function as a linear superposition of moments of the function. This perspective has applications in probability theory. The Laplace transform is invertible on a large class of functions. The inverse Laplace transform takes a function of a complex variable s (often frequency) and yields a function of a real variable t (time). Given a simple mathematical or functional description of an input or output to a system, the Laplace transform provides an alternative functional description that often simplifies the process of analyzing the behavior of the system, or in synthesizing a new system based on a set of specifications. So, for example, Laplace transformation from the time domain to the frequency domain transforms differential equations into algebraic equations and convolution into multiplication. It has many applications in the sciences and technology” (Korn & Korn 1967). More interesting information about the Fourier transform and the Laplace transform can be found, inter alia, in: (Phillips, Parr & Riskin 1995, Hilger 1999, Carlsson & Wittsten 2017).

As a transformation, the wavelet transform is similar to the Fourier. Both are based on the use of dot product operations of the test signal and the other part, which is known as the “kernel of the transformation”. The main difference between them is the kernel. The use of wavelets as the nucleus of the transformation makes it possible to present each continuous function with a certain accuracy expressed by wavelet coefficients.

In the Fourier transform the domain contains time functions and the co-domain contains frequency functions. However, the wavelet transform allows for the transition from a time-value system to a time-scale (frequency) system, which makes it possible to analyze the frequency change in the time domain. (It should be explained here that moving from the time-value system to the frequency-value system, we lose information about when a given event occurred). The formula for the Fourier transform is given by:

$$V(\omega) = \int_{-\infty}^{+\infty} v(t) e^{-j\omega t} dt. \quad (2)$$

There is one form of the Fourier transform for each category:
– continuous-time Fourier transform,
– continuous-time Fourier series,
– discrete-time Fourier transform,

The fast Fourier transform is a fast version of the discrete-time power signal and does not apply to any of the other transforms. The discrete-time power signal is defined by:

$$V(k) = \sum_{n=0}^{N-1} v(n) e^{-2j\pi nk/N}. \quad (3)$$

The above equation is really an N equation, one for each value of k (Mix & Olejniczak 2003, p. 19):

$$V(0) = \sum_{n=0}^{N-1} v(n) e^{0}, \quad (4)$$

$$V(2) = \sum_{n=0}^{N-1} v(n) e^{-2j\pi n/N}, \quad (5)$$

$$V(N-1) = \sum_{n=0}^{N-1} v(n) e^{-2j\pi n(N-1)/N}. \quad (6)$$

Again, the main difference between wavelet transform and Fourier transform is the kernel. For the kernel, Fourier transformation uses a sinusoidal function (i.e. periodic functions representing one frequency). However, in wavelet transform, the kernel is a wavelet, a special feature limited to certain requirements which must be met to be able to use it for the so-called multi-resolution analysis (e.g. it must have scaling function). There is an infinite number of such functions, and thus also an infinite number of wavelet transformations.

In the article I attempt to approximate a series of wavelets. Wavelets are basic functions for the wavelet transform (see Hadaś-Dyduch 2015, 2016a, 2016b, Hadaś-Dyduch, Balcerzak & Pietrzak 2016). As previously mentioned, the wavelet transform is a transformation similar to Fourier transform, thus "(...) wavelet coefficients can be calculated in the same way as Fourier coefficients, by using basis functions in an inner product calculation.

However, wavelets allow us to use an alternative scheme involving samples of the waveform supplied to a filter-down sample operation. For this, we must have an appropriate filter. Different wavelets are associated with different filters" (Mix & Olejniczak 2003, p. 24).

2. The Haar Wavelet

The Haar transform is one of several wavelet transforms that can be calculated with a formula. "In mathematics, the Haar wavelet is a sequence of rescaled «square-shaped» functions which together form a wavelet family or basis. Wavelet analysis is similar to Fourier analysis in that it allows a target function over an interval to be represented in terms of an orthonormal basis. The Haar sequence is now recognised as the first known wavelet basis and extensively used as a teaching example. (…) The Haar wavelet is also the simplest possible wavelet. The technical disadvantage of the Haar is that it is not continuous, and therefore not differentiable. This property can, however, be an advantage for the analysis of signals with sudden transitions, such as monitoring of tool failure in machines" (Lee & Tarng 1999, p. 241). The formula of Haar transform is given by:

$$c_{00} = \int_0^1 v(t)\, \varphi_{00}(t)dt, \tag{7}$$

$$c_{kj} = \int_0^1 v(t)\, v(t)\, \psi_{kj}(t)dt. \tag{8}$$

The function ϕ_{00} is called the scaling function, and the functions ψ_{kj} are called wavelets. The wavelet function corresponds to a bandpass filter (or a highpass filter). In contrast, the scaling function corresponds to a low-pass filter for approximation (averaging, smoothing the waveform). The scaling function is always assigned to one wavelet function (generating a family of scaling functions as for wavelet functions based on translation and scale).

"There are many wavelet basic functions other than Haar functions, and there is one wavelet transform for each set of basic functions. This is similar to the Fourier transforms, where there are four forms of the Fourier transform, but not similar in that there is a vast array of wavelet basic functions. Another important difference is in the way the coefficients for the wavelet transform are calculated. Most wavelet coefficients are calculated in a different way, using multirate sampling theory. There it is necessary to know only the filter coefficients. This method does not use the basic function" (Mix & Olejniczak 2003, p. 24).

Proposed in 1910 by the Hungarian mathematician A. Haar, the Haar transform is the simplest of the wavelet transforms, and one of the oldest transform functions. The Haar wavelet (Figure 1) is also the simplest possible wavelet.

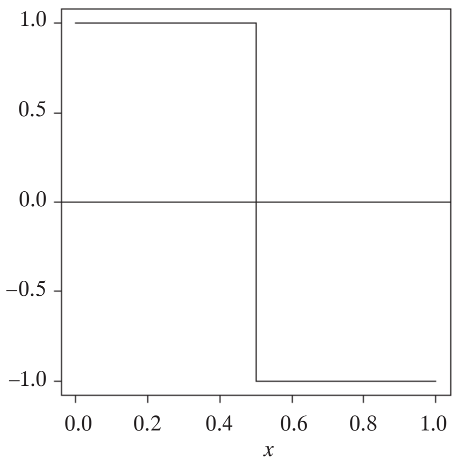

Fig. 1. Haar Wavelet

Source: the author's own elaboration.

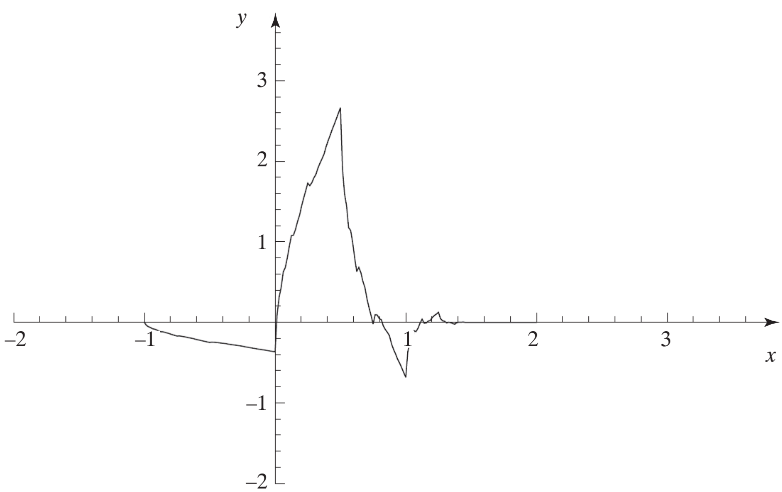

Fig. 2. Daubechies Wavelet

Source: the author's own elaboration.

Apart from the Haar wavelet, there are many other types of waves: Daubechies wavelets, Cohen-Daubechies-Feauveau wavelet, Mathieu wavelet, Legendre wavelet, Villasenor wavelet, to name just a few.

A special case of the Daubechies wavelet (Figure 2), the Haar wavelet, is also known as Db1. With each wavelet type of this class, there is a scaling function (called the father wavelet) which generates an orthogonal multiresolution analysis (see Daubechies 1992).

3. Model Specification

3.1. General Remarks

The aim of this article is to approximate and predict series with wavelets. It draws into one algorithm econometric methods with wavelet analysis. Econometric methods and wavelet transform are combined for the construction of a model that predicts a time series.

The choice of econometric methods for prediction is wide. There are two groups of econometric methods involved: static spatial information system and dynamic system of spatial information.

A static spatial information system presents the relationship between the signal input and output circuit. The relationship is usually constant, but depends on the time. Examining time characteristics in the context of the static does not make sense, because they do not say anything about a system in which there are no state variables. By contrast, a dynamic spatial information system accounts for the position of the object and its inception. This makes it possible to view the changes over time in the maps generated by the system. A geographic information system acquires, processes and shares data containing spatial information and accompanying descriptive information about the objects featured in the portion of the space covered by the system's operation.

From among the available and well-know econometric methods, for this article I chose only one group – adaptive methods. The difference between conventional methods and methods of adaptation include:

1) classical methods:

– stimulus is often far from the threshold,

– stimulus values to be presented are fixed before the experiment;

2) adaptive methods:

– modifications of the method of constant stimuli and method of limits,

– stimulus values to be presented depend critically on the responses that preceded them.

3.2. Algorithm Specification

My algorithm can be presented in the following main stages:

1. A one-dimensional time series is divided into smaller, equinumerous units, keeping the chronology of time.

2. For each series resulting from the division series in the base point 1, we determine the coefficients of wavelet for the following equations:

$$\begin{cases} \sum_{n=0}^{L-1} h_n & = \sqrt{2}, \\ \sum_{n=0}^{L-1} h_n h_{n+2m} & = \delta_m \\ \sum_{q=0}^{L-1} q^k (-1)^k h_{L-1-q} & = 1, \end{cases} \tag{9}$$

where L is filter length, and:

$$\delta_n = \begin{cases} 0 & \text{for} \quad n \neq 0 \\ 1 & \text{for} \quad n = 0 \end{cases}. \tag{10}$$

Signal processing using wavelet transform uses filters. Filter h is called a low-pass filter, which is defined as:

$$h_n = \langle \phi(x), \phi_{1,n}(x) \rangle, \tag{11}$$

$$\phi(x) = \sqrt{2} \sum_{n \in Z} h_n \cdot \phi(2x - n). \tag{12}$$

3. Determine the function approximating each series according to the formula:

$$f(x) = \sum_{l \in Z} c_{j-1,l} \phi_{j-1,l}(x) \sum_{l \in Z} d_{j-1,l} \psi_{j-1,l}(x), \tag{13}$$

where:

$$c_{j,n} = \langle f_j(x), \phi_{j,n}(x) \rangle, \tag{14}$$

$$d_{j,n} = \langle f_j(x), \psi_{j,n}(x) \rangle, \tag{15}$$

$$\psi(x) = \sqrt{2} \sum_{n \in Z} g_n \phi(2x - n). \tag{16}$$

4. Construction of models segmented according to the initial division of the unit series of the base.

5. Determination of theoretical values arising from the specific functions and series unit.

6. The calculation of the final value of the theoretical forecasted variable according to the formula:

$$\hat{f}_t = \frac{1}{k_i}\sum_{j=1}^{k}\hat{f}_{ij}(t), \qquad (17)$$

where:

$\hat{f}_{ij}(t)$ is the final theoretical value for period or moment t,

k_i is the number of "segments" of theoretical variable values for the period or the moment t.

7. The solution to the problem:

$$Min\left\{\sqrt{\frac{1}{n}\sum_{t=1}^{n}((a\hat{y}_t+(1-\alpha)y_{t-1})-y_t)^2}\right\} \qquad (18)$$

on the assumption $\alpha \in \langle 0,1 \rangle$.

8. Prediction errors.

The article is used to approximate Daubechies wavelet. In contrast to Haar's simple-step wavelets, which exhibit jump discontinuities, Daubechies wavelets are continuous. As a consequence of their continuity, Daubechies wavelets approximate continuous signal more accurately with fewer wavelets than do Harr's wavelets, but require intricate algorithms based upon a sophisticated theory. The Daubechies wavelets are a family of orthogonal wavelets characterised by the maximal number of vanishing moments for some given support. With each wavelet type of this class, there is a scaling function which generates an orthogonal multiresolution analysis. Furthermore, each Daubechies wavelet is compactly supported. The Daubechies wavelets are neither symmetric nor antisymmetric around any axis, except for Db1, which is in fact the Haar wavelet. It is not possible to satisfy symmetry conditions given all the other properties of the Daubechies wavelets (see Daubechies 1992).

The Daubechies wavelets begin by approximating the samples by the scaling function of the multiples of shifted basic building blocks:

$$\tilde{f}(r) = a_{-2}\phi(r+2) + a_{-1}\phi(r+1) + a_0\phi(r) + \ldots + a_2n_{-1}\phi(r-[2^n-1]), \qquad (19)$$

where:

$$\psi(r) = -\frac{1+\sqrt{3}}{4}\phi(2r-1) + \frac{3+\sqrt{3}}{4}\phi(2r) - \frac{3-\sqrt{3}}{4}\phi(2r+1) + \frac{1-\sqrt{3}}{4}\phi(2r+2), \qquad (20)$$

$$\psi(r) = 0 \text{ for } r < -1 \text{ or } r > 2, \qquad (21)$$

$$\phi(r) = \frac{1+\sqrt{3}}{4}\phi(2r) + \frac{3+\sqrt{3}}{4}\phi(2r-1) + \frac{3-\sqrt{3}}{4}\phi(2r-2) + \frac{1-\sqrt{3}}{4}\phi(2r-3), \quad (22)$$

$$\sum_{k \in Z} \phi(k) = 1, \quad (23)$$

$$\varphi(r) = 0 \text{ for } r \leq 0 \vee r \geq 3, \quad (24)$$

$$D_j = \{k2^j : k \in Z\}, D = \bigcup_{j \in Z} D_j = \bigcup_{j=0}^{\infty} D_j. \quad (25)$$

Shifts $\varphi(r - l)$ or $\varphi(r)$ by integers $l < -2$ or $l > 2^n - 1$ equal zero where $0 \leq r \leq 2^n$, and consequently do not affect the approximation of $\hat{f}$.

Here are the following steps to find approximation (see Mix & Olejniczak 2003):

1. Start at the beginning of the waveform and compare to wavelet by correlation.
2. Shift the wavelet to the right and repeat step 1. Do this until you have covered the entire signal.
3. Scale the wavelet and repeat steps 1 and 2.
4. Repeat steps 1 through 3 for all scales.

By a combination $\hat{f}$ of shifted building blocks φ and wavelets ψ, Daubechies wavelet can approximate a function f, which may represent any signal. A simple and common choice of the coefficient a_k consists in setting, for each $k \in \{0, \ldots, 2^n - 1\}$, $a_k := s_k$, the corresponding approximation (see Yves 1999):

$$\tilde{f} = \sum_{k=0}^{2^n-1} s_k \varphi(\mathrm{r} - k) \quad (26)$$

nearly interpolates f at the sample points $s_k = f(k)$.

4. Data to Be Supplied to the Algorithm

The research was based on a series of financial exchange rates established by the National Bank of Poland. Quotations of the value of three currencies' (Czech crown, Romanian leu and euro) were from the period 1st January 2000 to 31st May 2016. As Figures 3, 4 and 5 show, the exchange rates varied widely in the period considered.

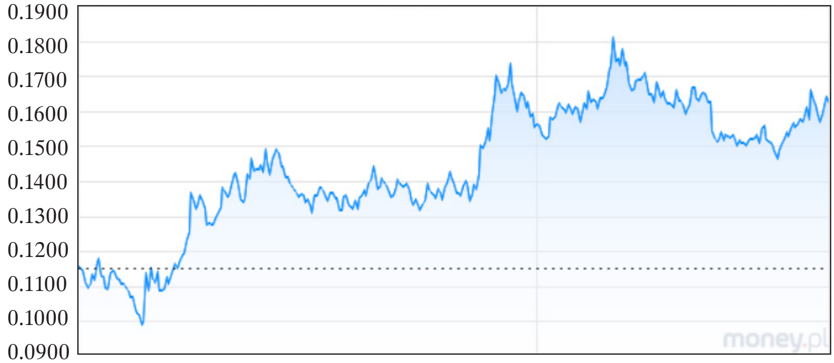

Fig. 3. Chart for the Czech Crown in the Period 2000-01-01 to 2016-05-31

Source: http://www.money.pl/pieniadze/nbparch/srednie/. Accessed: 1 June 2016.

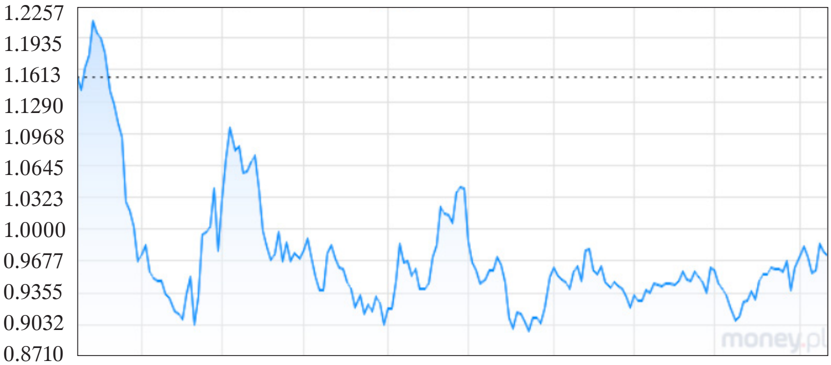

Fig. 4. Chart for the Romanian Leu in the Period 2000-01-01 to 2016-05-31

Source: http://www.money.pl/pieniadze/nbparch/srednie/. Accessed: 1 June 2016.

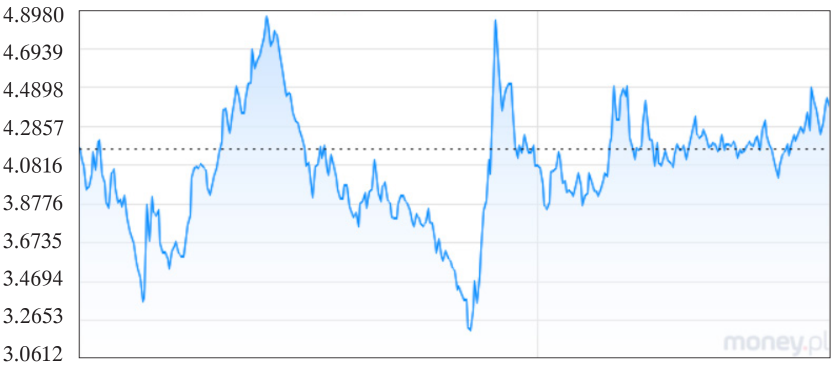

Fig. 5. Chart for the Euro in the Period 2000-01-01 to 2016-05-31

Source: http://www.money.pl/pieniadze/nbparch/srednie/. Accessed: 1 June 2016.

5. Results and Discussion

Prediction series for one period forward using wavelets provided fairly good results, which is best illustrated by the results quoted below: The *RSME* error was 0.98% for the Czech crown, 0.87% for the Romanian leu and 0.79% for the euro.

The value of the prediction error depends on many factors, among them the method used to expand the series input data to calculate the wavelet coefficients. In the results cited above, the polynomial method is:

$$p(r) = p_0 + p_1(r - [2^n - 1]) + p_2(r - [2^n - 1])(r - [2^n]) + \\ + p_3(r - [2^n - 1])(r - [2^n])(r - [2^{n+1} - 1]). \qquad (27)$$

For comparison, Table 1 shows the results obtained by using other methods. The following methods (assuming that the initial series has the form: $p_0, p_1, p_2, \ldots, p_{2^n-2}, p_{2^n-1}$):

– method 1:

$$\underbrace{0, 0, 0, \cdots, 0}_{\text{extension}}, \underbrace{p_0, p_1, p_2, \cdots, p_{2^n-2}, p_{2^n-1}}_{\text{series}}, \underbrace{0, 0, 0, \cdots, 0}_{\text{extension}}, \qquad (28)$$

– method 2:

$$\underbrace{p_{2^n-1}, \cdots, p_0}_{\text{extension}}, \underbrace{p_0, p_1, p_2, \cdots, p_{2^n-2}, p_{2^n-1}}_{\text{series}}, \underbrace{p_{2^n-1}, \cdots, p_0}_{\text{extension}}, \qquad (29)$$

– method 3:

$$\underbrace{p_0, p_1, p_2, \cdots, p_{2^n-2}, p_{2^n-1}}_{\text{extension}}, \underbrace{p_0, p_1, p_2, \cdots, p_{2^n-2}, p_{2^n-1}}_{\text{series}}, \underbrace{p_0, p_1, p_2, \cdots, p_{2^n-2}, p_{2^n-1}}_{\text{extension}}, \qquad (30)$$

– method 4:

$$\underbrace{p_0, p_1, p_2, \cdots, p_{2^n-2}, p_{2^n-1}}_{\text{series}}, \underbrace{p_{2^n-1}, \cdots, p_0}_{\text{extension}}, \underbrace{p_0, p_1}_{\text{short extension}}. \qquad (31)$$

The need to extend the series input data to determine the wavelet coefficients appears in the case of filters. Length L is greater than 2. This follows from the fact that the calculation of the wavelet coefficients of expansion for the last element of the finite signal filter should, in theory, go beyond the signal. However, this did not occur.

Depending on the method used to extend the series, various prediction errors can be produced.

Table 1. Results

Currency	The method of extending the series				
	I	II	III	IV	Polynomial
Czech crown	2.21%	2.10%	1.75%	1.61%	0.98%
Romanian leu	2.51%	2.25%	1.84%	1.31%	0.87%
Euro	3.1%	2.95%	2.01%	1.11%	0.79%

Source: the author's own calculations.

For this study I have used Daubechies wavelet. The study I have described in this article could be used as an introduction to further research, with subsequent algorithms being built upon the ones reported here. Innovative algorithms will be extended by different algorithms, including those presented in (Biernacki 2007, 2009). Other analyses of wavelet analysis are presented, among others, in (Hadaś-Dyduch 2015, 2016b, 2016c), which also presents comparative analyses and interesting conclusions.

7. Conclusions

The article has described the series prediction and approximation using wavelet. The research was based on Daubechies wavelet. Daubechies wavelets, based on the work of I. Daubechies, are a family of orthogonal wavelets defining a discrete wavelet transform and characterized by a maximal number of vanishing moments for some given support. While Daubechies wavelets were used for this study, other wavelets including the Meyer, Morlet, Haar or "Mexican hat" can all be used. Wavelet analyses must have finite energy and an average value of zero. As a result, they take the form of short-term oscillations.

This article has not compared the results with other prediction models, because the purpose of the study was not to evaluate and select the best model prediction, but to present my own model for prediction and presentation approximation of financial time series using wavelets.

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Abstract

Aproksymacja szeregów czasowych z falkami

Finansowe szeregi czasowe wykazują charakterystyczne własności. Wśród nich można wymienić m.in.: występowanie zjawiska grupowania wariancji, leptokurtyczność rozkładów stóp zwrotu (tzw. grube ogony rozkładu) oraz ujemną korelację pomiędzy stopami zwrotu a zmiennością ich wariancji. Zjawiska te powodują, że w wielu przypadkach stosowanie standardowych metod estymacji parametrów i prognozowania nie przynosi zadowalających rezultatów. Ważną cechą finansowych szeregów czasowych jest fakt, że szeregi finansowe charakteryzują się długimi próbkami, co powoduje, że stosowane do ich estymacji modele mogą być bardziej rozbudowane.

Celem artykułu jest aproksymacja i predykcja szeregów finansowych z falkami z uwzględnieniem tzw. efektów brzegowych. W artykule opisano autorski model prognozowania finansowych szeregów czasowych oraz przedstawiono podstawowe informacje o falkach niezbędne do właściwego zrozumienia proponowanego algorytmu falkowego. W autorskim algorytmie wykorzystano falkę Daubechies.

Słowa kluczowe: predykcja, falki, transformata falkowa, aproksymacja.